Math 132: Differential Topology

§ Orientation

Def
Suppose V is a finite dim'l real vector space,
and $\beta = \{v_1, \cdots, v_k\}$ and $\beta' = \{v_1', \cdots, v_k'\}$ are two ordered basis.
Then there is a unique linear isomorphism $A : V \to V$ such that $\beta' = A\beta$.
We say β and β' are equivalently oriented if $\det A > 0$.

This gives an equivalence relation on the set of all ordered basis of V,
and there are exactly two equivalence classes ($GL(V)$ has 2 components).

Def An orientation of V is a choice of one of those two equivalance classes.

An orientation induces a sign ($+$ or $-$) on any ordered basis of V.
* When $V=0$, we separately define its orientation to be a choice of a sign $\in \{\pm\}$.

Ex Say, $V = \mathbb{R}^2$ and choose $[(e_1, e_2)]$ for the orientation.

Then

, ... are positively oriented,

while

, ... are negatively oriented

Note, if V and W are oriented real vector spaces of the same dimension, then any isomorphism $A : V \to W$ either preserves or reverses orientation.

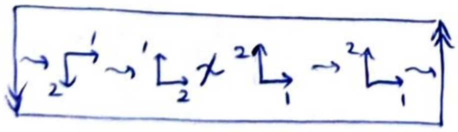
Def An orientation of M, a manifold with boundary, is a smooth choice of orientations for all tangent spaces T_pM.

i.e. around each $p \in M$, there is a local parametrization $h : U \to M$, $U \subset H^m \subset \mathbb{R}^m$, such that $dh_u : T_uU \cong \mathbb{R}^m \to T_{h(u)}M$ preserves orientation for all $u \in U$, where $\mathbb{R}^m$ is given the standard orientation $[(e_1, \cdots, e_m)]$.

A smooth map between oriented mflds of same dim, whose derivative preserves orientations at every point is called an orientation-preserving map.

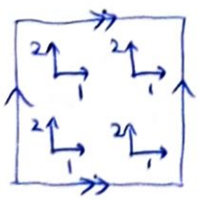
Ex Not all manifolds can be oriented!

Möbius band



$\rightsquigarrow$ not orientable.

Ex T^2



S^2

$\rightsquigarrow$ orientable

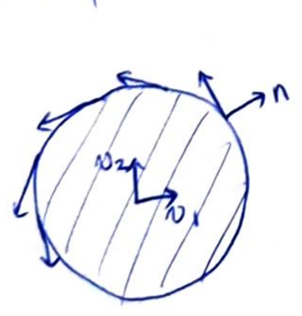
<u>Prop</u> A <u>connected</u>, orientable manifold with boundary admits exactly two orientations.

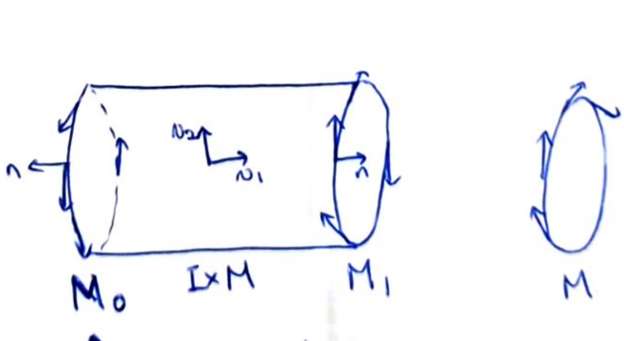
proof) The set of points at which two orientations agree and the set where they disagree are both open. Consequently, two orientations of a connected manifold are either identical or opposite. ∎

<u>Rmks:</u>

- The product $M \times N$ of two oriented manifolds (where one of them is boundaryless) acquires a <u>product orientation</u>, for if α and β are ordered basis for T_pM and T_qN, then $(\alpha \times 0, 0 \times \beta)$ is an ordered basis for $T_{(p,q)}(M \times N) \cong T_pM \times T_qN$.
- An orientation of M induces a <u>boundary orientation</u> on ∂M; this can be done by declaring the sign of any ordered basis $\beta = \{v_1, \cdots, v_{m-1}\}$ of $T_p(\partial M)$ to be the sign of the ordered basis $\{n_p, \beta\} = \{n_p, v_1, \cdots, v_{m-1}\}$, where n_p is the <u>outward unit normal</u> at p.

<u>Ex</u>





note, opposite orientation!

$\Rightarrow \partial(I \times M) = M_1 - M_0$

If $V = V_1 \oplus V_2$, the orientations on any two of V, V_1, V_2 induces a direct sum orientation on the third. (The ordering of summands is important!)

We can use this to orient $Q = f^{-1}(P)$ if $f: M \to N \supset P$, $f \pitchfork P$, $f|_{\partial M} \pitchfork P$

and M, N, P are all oriented.

Let $N_x(Q;M)$ be the orthogonal complement of T_xQ in T_xM

so that $T_xM = \boxed{N_x(Q;M)} \oplus T_xQ$. $\rightsquigarrow$ orientation on T_xQ!

By transversality, $T_yN = \boxed{df_x(N_x(Q;M))} \oplus T_yP$.

oriented

Prop $\partial f^{-1}(P) = (-1)^{\operatorname{codim} P} (f|_{\partial M})^{-1}(P)$.

proof sketch) $T_xM = N_x(Q;M) \oplus \underbrace{\mathbb{R}\cdot n_x}_{\text{outward normal to } \partial Q} \oplus T_x(\partial Q)$

$= \mathbb{R}\cdot n_x \oplus N_x(Q;M) \oplus T_x((f|_{\partial M})^{-1}(P))$

$\Rightarrow$ the orientations differ by $(-1)^{\operatorname{codim} Q} = (-1)^{\operatorname{codim} P}$ ∎